INTEGRATING RATIONAL FUNCTIONS OF THE FORM $\frac{ax+b}{cx+d}$

Aim: To rewrite $\int \frac{ax+b}{cx+d}\,dx$ to the form $\int \frac{[\ \ ](cx+d)\ +\ [\ \ ]}{cx+d}\,dx$

Examples: Integrate these functions:

a) $\int \frac{2x+3}{x+2}\,dx$

$2x+3 = [\ \ ](x+2) + [\ \ ] = 2(x+2) + [\ \ ] = 2(x+2) + [-1] = 2(x+2) - 1$

$\int \frac{2x+3}{x+2}\,dx = \int \frac{2(x+2)-1}{x+2}\,dx = \int \frac{2(x+2)}{(x+2)} - \frac{1}{(x+2)}\,dx = \int 2 - \frac{1}{(x+2)}\,dx$

$\int 2 - \frac{1}{(x+2)}\,dx = 2x - \ln|x+2| + c.$

b) $\int \frac{3x-5}{x+1}\,dx$

$3x-5 = [\ \ ](x+1) + [\ \ ] = 3(x+1) + [\ \ ] = 3(x+1) + [-8] = 3(x+1) - 8$

$\int \frac{3x-5}{x+1}\,dx = \int \frac{3(x+1)-8}{x+1}\,dx = \int \frac{3(x+1)}{(x+1)} - \frac{8}{(x+1)}\,dx = \int 3 - \frac{8}{(x+1)}\,dx$

$\int 3 - \frac{8}{(x+1)}\,dx = 3x - 8\ln|x+1| + c.$

c) $\int \frac{3-2x}{1-x}\,dx$

$3-2x = [\ \ ](1-x) + [\ \ ] = 2(1-x) + [\ \ ] = 2(1-x) + 1$

$\int \frac{3-2x}{1-x}\,dx = \int \frac{2(1-x)+1}{1-x}\,dx = \int \frac{2(1-x)}{(1-x)} + \frac{1}{(1-x)}\,dx = \int 2 + \frac{1}{(1-x)}\,dx$

$\int 2 + \frac{1}{(1-x)}\,dx = 2x + {}^{-}1\ln|1-x| + c = 2x - \ln|1-x| + c.$

d) $\int \frac{x}{3x-2}\,dx$

$x = [\ \ ](3x-2) + [\ \ ] = \frac{1}{3}(3x-2) + [\ \ ] = \frac{1}{3}(3x-2) + \frac{2}{3}$

$\int \frac{x}{3x-2}\,dx = \int \frac{{}^{1}/_{3}(3x-2) + {}^{2}/_{3}}{3x-2}\,dx = \int \frac{{}^{1}/_{3}(3x-2)}{(3x-2)} + \frac{{}^{2}/_{3}}{(3x-2)}\,dx = \int \frac{1}{3} + \frac{{}^{2}/_{3}}{(3x-2)}\,dx$

$\int \frac{1}{3} + \frac{{}^{2}/_{3}}{(3x-2)}\,dx = \frac{1}{3}x + \frac{2}{3}\cdot\frac{1}{3}\cdot\ln(3x-2) + c = \frac{1}{3}x + \frac{2}{9}\ln(3x-2) + c.$

Examples where the numerator and/or denominator

are not linear:

e) $\int \frac{x^2-x+8}{x+3}\, dx$

$$
\begin{array}{r}
x-4 \\
x+3\overline{)\,x^2-x+8} \\
\underline{x^2+3x} \\
-4x+8 \\
\underline{-4x-12} \\
20
\end{array}
$$

Do polynomial long division to get: $\frac{x^2-x+8}{x+3} = x - 4 + \frac{20}{x+3}$.

So $\int \frac{x^2-x+8}{x+3}\, dx = \int x - 4 + \frac{20}{x+3}\, dx = \frac{x^2}{2} - 4x + 20\ln|x+3| + c$.

f) $\int \frac{x^3+2x^2-8}{x^2}\, dx$

$\int \frac{x^3+2x^2-8}{x^2}\, dx = \int \frac{x^3}{x^2} + \frac{2x^2}{x^2} - \frac{8}{x^2}\, dx = \int x + 2 - 8x^{-2}\, dx = \frac{x^2}{2} + 2x - \frac{8x^{-1}}{-1} + c$

Delta Ex 17.5 pg 172